Exactly solvable models and investigation of level crossing

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A large class of exactly solvable two-dimensional models is investigated on the basis of the inverse scattering problem in the adiabatic approach. Exactly solvable models with prescribed spectral properties are constructed by using a generalized technique of Bargmann potentials. The matrix elements of the operator of the connection and the scalar potential are generated and studied in the vicinity of level crossings for terms with a given functional dependence. It is shown that in a number of cases the potentials, the basis functions corresponding to them, and the connections are singular at the degeneracy points of the wave states. The problem of level crossing is studied by exactly solvable models of the inverse problem. © *1996 American Institute of Physics.* [S1063-7796(96)00204-5]

1. INTRODUCTION

A specific feature of complex systems such as molecules, atoms, and nuclei is that they are characterized by several degrees of freedom that, as a rule, cannot be separated in strongly interacting systems. The adiabatic representation[1–3] makes it possible to take into account the mutual influence of a slowly varying external (collective) field and a rapidly varying internal (intrinsic) field. Since the first studies of Born,[1] the direct scattering problem in the adiabatic approach has had a fairly rich historical development. However, a constructive approach to the three-particle inverse scattering problem was proposed relatively recently.[4,5] It is based on a global adiabatic representation for the three-particle wave function constructed in terms of local adiabatic expansions of the Faddeev components.[6,7]

One of the main advantages of the inverse scattering method is the possibility of obtaining analytic solutions in both linear and nonlinear wave theory. Modeling of quantum two-, three-particle, and effective many-particle composite systems is also a very important application of exactly solvable models. In its turn, the inverse problem in the adiabatic representation extends the possibilities of the inverse scattering method in the investigation of many real quantum systems with complicated dynamics. The topological nature of the space of composite systems is determined by the interaction of the collective and intrinsic fields, which can lead to the occurrence of monopole gauge potentials and, as a consequence, to phenomena such as nonintegrable geometric phases,[8–10] which were discovered by Berry,[8] the molecular Aharonov–Bohm effect,[11] chaos for collective motion,[12] and a fractional statistics effect.[13,14] In our view, exactly solvable models in the adiabatic approach make it possible in a number of cases to model these processes, and they serve as a good formalism for investigating monopole gauge potentials in quantum systems and the related problem of level crossing. In connection with the intensive investigation of these effects,[8–19] the old problem of level crossing associated with Landau–Zener transitions[20,21] and its connection with topological effects in quantum systems becomes especially topical. Here we propose to investigate the problem of level crossing on the basis of a large class of exactly solvable models in the inverse scattering approach in the adiabatic representation.

The multidimensional inverse problem in the adiabatic approach, which in its mathematical essence is one of the forms of dimensional reduction of a space $M=B\times\hat{M}$, reduces to the mutually consistent solution of two inverse problems in the spaces B and $\hat{M}$, which have lower dimension than the original space (Refs. 4, 5, 22, and 23). One of them is a parametric problem and is formulated for a Schrödinger equation that describes the fast dynamics in its parametric dependence on the slow variables, while the other is a many-channel problem for a system of equations of gauge type that describes the slow motion. On the basis of a generalization of the technique of Bargmann potentials, used both for the parametric Schrödinger equation and for the many-channel system of gauge equations, a method was developed in Refs. 5 and 22–24 for constructing a large class of exactly solvable models in a space of several dimensions; the method was initially proposed in Ref. 22. Here we investigate two-dimensional exactly solvable models in a mutually consistent formulation of the inverse problems for both of these cases, and we give some examples in illustrations that demonstrate the possibilities of the inverse scattering method. In particular, we give a graphical representation of two-dimensional potentials and basis functions for systems with one, two, and three bound states in the regular and singular cases. We investigate a number of two-dimensional exactly solvable models constructed on the basis of the parametric inverse problem in the Gel'fand–Levitan and Marchenko approaches with a given dependence of the spectral characteristics on the external coordinate variable.

Using the functions of the parametric problem, which are obtained in closed analytic form, we develop a method for determining the exchange interaction induced by these functions and, on this basis, a method for calculating the amplitudes for transitions between levels. We investigate the problem of level crossing and consider some geometrical aspects of the adiabatic approach that are associated with the occurrence of monopole gauge potentials in the presence of level degeneracy.

Such an approach can be regarded as a useful tool for investigating the problem of Landau level crossing and some

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other problems associated with the topological nature of the space of composite systems with several degrees of freedom. At the same time, possibilities are opened up for investigating the effect of the fast dynamics of individual particles on the collective dynamics of the system as a whole and, conversely, the influence of the collective dynamics on the intrinsic two- and three-particle interactions in the system.

2. GEOMETRIC PHASES

We assume that the state $|\Psi(t)\rangle$ of the system evolves in accordance with the Schrödinger equation

$$i\hbar \frac{d|\Psi(t)\rangle}{dt} = H(t)|\Psi(t)\rangle. \tag{1}$$

In quantum mechanics, it is, in the general case, a difficult problem to solve Eq. (1), since in the presence of an energy dependence of the Hamiltonian the energy is not conserved and stationary states do not exist. We introduce for the wave function a unitary transformation $C(t)$:

$$|\Psi(t)\rangle = C(t)|\psi(t)\rangle, \tag{2}$$

which is such that in the equation obtained from (1) for $|\psi(t)\rangle$,

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = h|\psi(t)\rangle, \tag{3}$$

the Hamiltonian

$$h = C^{\dagger}HC - i\hbar C^{\dagger}\dot{C} \tag{4}$$

does not depend on the time, i.e., $(\partial/\partial t)h = 0$. From this, we obtain an equation for the transformation $C(t)$:

$$\dot{C}^{\dagger}HC + C^{\dagger}\dot{H}C + C^{\dagger}H\dot{C} - i\hbar\dot{C}^{\dagger}\dot{C} - i\hbar C^{\dagger}\ddot{C} = 0, \tag{5}$$

where we have used the notation $\dot{C} = (d/dt)C$. The wave function of Eq. (3) can be represented in the form

$$|\psi(t,\mathbf{X})\rangle = \exp\left(\frac{i}{\hbar}\int_0^t E(t')dt'\right|\left.\phi(\mathbf{X})\right\rangle \tag{6}$$

and, essentially, reduces to the solution of the problem

$$h|\phi(\mathbf{X})\rangle = E|\phi(\mathbf{X})\rangle \tag{7}$$

with time-independent Hamiltonian h.

For a self-adjoint operator h, the functions $|\phi(\mathbf{X})\rangle$ form a complete and orthonormal set. Substituting $|\psi(t,\mathbf{X})\rangle$ in (3) and using the orthogonality of the functions $|\phi(\mathbf{X})\rangle$, we obtain, with allowance for (4),

$$\frac{iE(t)}{\hbar} = -\frac{i}{\hbar}\langle\Psi(t)|H(t)|\Psi(t)\rangle\langle\phi|C^{\dagger}\dot{C}|\phi\rangle. \tag{8}$$

With allowance for this relation in (2), the wave function of Eq. (1) becomes

$$|\Psi(t)\rangle = C(t)|\phi\rangle\exp\left(-\frac{i}{\hbar}\int_0^t\langle\Psi(t')|H(t')|\Psi(t)\rangle dt' - \int_0^t\langle\phi|C^{\dagger}\dot{C}|\phi\rangle dt'\right). \tag{9}$$

The first term in the brackets is the dynamical part of the phase, and the second term is the geometric part. Therefore, if we eliminate the dynamical part from the phase factor,[10,18] the wave function $|\Psi'(t)\rangle$ of Eq. (1) can be rewritten as

$$|\Psi'(t)\rangle = C(t)|\phi\rangle\exp\left(-\int_0^t\langle\phi|C^{\dagger}\dot{C}|\phi\rangle dt'\right). \tag{10}$$

The phase difference between $|\Psi'(0)\rangle$ and $|\Psi'(T)\rangle$ is the geometric phase δ, which is determined as follows. We multiply the relation (10) with $t=T$ by the state vector (10) at the initial time, $\langle\Psi'(0)|$:

$$\begin{aligned}\langle\Psi'(0)||\Psi'(T)\rangle &= \langle\phi|C^{\dagger}(0)C(T)|\phi\rangle \\ &\quad\times\exp\left(-\int_0^T\langle\phi|C^{\dagger}\dot{C}|\phi\rangle dt'\right) \\ &= \rho\,\exp(i\delta).\end{aligned} \tag{11}$$

Since the factor multiplying the exponential is a complex number,

$$\begin{aligned}\langle\phi|C^{\dagger}(0)C(T)|\phi\rangle &= \rho\,\exp(i\mathrm{Arg}[\langle\phi|C^{\dagger}(0)C(T)|\phi\rangle]) \\ &= \rho\,\exp(i\beta),\end{aligned}$$

the phase δ can be represented in the form

$$\delta = \beta + i\left(-\int_0^T\langle\phi|C^{\dagger}\dot{C}|\phi\rangle dt'\right). \tag{12}$$

If in the process of evolution of the quantum system up to some time $t=T'$ we have $C(T')|\phi\rangle = C(0)|\phi\rangle$, then $\beta=0$, and the phase (12) is exactly the Aharonov–Anandan geometric phase.[10]

In quantum mechanics, the use of unitary transformations of the type (4) is a standard technique. It is effectively used in obtaining the geometric phases[8] in the case of a slow and smooth cyclic evolution in time of the collective coordinates $\mathbf{x}(t)\in\mathsf{B}$ of the system governed by the Hamiltonian $H(t)\equiv H(\mathbf{x}(t))$. Then the solution of Eq. (1) is sought in the form of the expansion

$$\begin{aligned}&|\Psi(t,\mathbf{x}(t),y)\rangle \\ &= \sum_n c_n(t)\exp\left(\frac{-i}{\hbar}\int_0^t\mathscr{E}_n(\mathbf{x}(t'))dt'\right)|\phi_n(\mathbf{x}(t);y)\rangle\end{aligned} \tag{13}$$

with respect to a complete orthonormal set of eigenfunctions $\{|\phi_n(\mathbf{x}(t);y)\rangle\}$ of the instantaneous Hamiltonian

$$H(\mathbf{x}(t))|\phi_n(\mathbf{x}(t);y)\rangle = \mathscr{E}_n(\mathbf{x}(t))|\phi_n(\mathbf{x}(t);y)\rangle. \tag{14}$$

Substituting the expansion (13) in Eq. (1) and using the orthonormality $\langle n|m\rangle = \delta_{nm}$ of the basis functions $|\phi_n(t)\rangle \equiv |n(t)\rangle$, we can readily obtain a system of linear differential equations for the coefficients $c_n(t)$:

$$\begin{aligned}i\hbar\frac{dc_n(t)}{dt} &= \left\{-i\hbar\sum_m\langle m|\dot{n}\rangle c_m(t) + \sum_m\langle m|H|n\rangle c_m(t)\right\} \\ &\quad\times\exp\left[-\frac{i}{\hbar}\int_0^t(\mathscr{E}_n(t') - \mathscr{E}_m(t'))dt'\right] \\ &\quad - \mathscr{E}_n(t)c_n(t).\end{aligned} \tag{15}$$

The system of equations for $c(t)$ with allowance for (14) can be rewritten in the form

$$\dot{c}_n(t)=\sum_m B_{nm}(\mathbf{x}(t))\exp\left[-\frac{i}{\hbar}\int_0^t(\mathscr{E}_n(t')-\mathscr{E}_m(t'))dt'\right]c_m(t). \tag{16}$$

Here the matrix elements of the exchange interaction,

$$B_{nm}(\mathbf{x}(t))=\langle n|\dot{m}\rangle=A_{nm}(\mathbf{x})\cdot\dot{\mathbf{x}}(t),$$

where

$$A_{nm}(\mathbf{x})=\langle n(\mathbf{x})|\nabla_{\mathbf{x}}|m(\mathbf{x})\rangle \tag{17}$$

are generated by the basis functions $|n\rangle$ of the "instantaneous" Hamiltonian (14).

Differentiating Eq. (14) with respect to t, we obtain for $n\neq m$ the relation

$$\langle m|\dot{n}\rangle=\frac{\langle m|\dot{H}|n\rangle}{\mathscr{E}_n-\mathscr{E}_m}.$$

The adiabatic approximation corresponds to the fact that the admixture of other states with the given $|n\rangle$ is small, i.e., the estimate of the nondiagonal elements of the relation (16) leads to the condition

$$\frac{\langle m|\dot{H}|n\rangle}{(\mathscr{E}_n-\mathscr{E}_m)^2}\ll 1, \tag{18}$$

and when this condition is satisfied the system will remain to a good approximation in the same state as it evolves. As follows from the relation (15), this gives

$$c_n(t)=\exp\left[-\int_0^t\langle n|\dot{n}\rangle dt'\right]c_n(0).$$

Berry showed that for cyclic adiabatic evolution the phase

$$\delta=-i\int_0^T\langle n|\dot{n}\rangle dt'$$

does not depend on the choice of the state $|n(t)\rangle$ and, therefore, reflects a geometric property of the parameter space $\mathbf{x}\in\mathsf{B}$, on which the Hamiltonian $H(\mathbf{x}(t))$ depends. Indeed, with allowance for (17) the final relation can be rewritten in the form

$$\delta=-i\oint_\Gamma\langle n(\mathbf{x})|\nabla_{\mathbf{x}}|n(\mathbf{x})\rangle\cdot\dot{\mathbf{x}}(t)$$

$$=-i\ \mathrm{Im}\int\int_\Gamma d\mathbf{S}\cdot\mathrm{curl}\ \mathbf{A}. \tag{19}$$

Here $\mathbf{S}$ is a surface bounded by the contour Γ. Berry's conclusion is based on the fact that the curl of the vector $\langle n(\mathbf{x})|\nabla_{\mathbf{x}}|n(\mathbf{x})\rangle$ does not depend on the choice of the phase of the eigenstate $|n(\mathbf{x})\rangle$. Indeed, if the vector $|n(\mathbf{x})\rangle$ is replaced by the vector $|n(\mathbf{x})\rangle\to\exp i\mu(\mathbf{x})|n(\mathbf{x})\rangle$, then $\langle n|\nabla|n\rangle\to\langle n|\nabla|n\rangle+i\nabla\mu$, and at the same time the curl of the vector $\mathbf{A}$ is unchanged. Thus, the vector need not be unique, but the curl of the connection, which determines the geometric phase, is.

It is obvious that the representation (13) is restricted and does not cover some situations, namely, when there is no slow evolution of the Hamiltonian $H(t)\equiv H(\mathbf{x}(t))$ with the time and, in a second case, which is not always identical to the first, when it is not possible to ignore the Hamiltonian that governs the collective "slow" degrees of freedom of the system compared with the Hamiltonian that describes the intrinsic interparticle interaction. The geometric phase (12) introduced by Aharonov and Anandan[10] is valid for any cyclic evolution and in the special case of adiabatic evolution contains the Berry phase.

When levels approach each other, the condition of adiabaticity is violated, as is readily seen from (18). In this situation, diabatic dynamics is usually adopted—the theory of Landau–Zener transitions is used. In their studies of Refs. 12 and 25, Bulgac and Kuznezov investigate the mutual influence of the collective and intrinsic degrees of freedom and note that it is manifested in macroscopic processes such as dissipation, collective diffusion, chaotic behavior of both slow and fast degrees of freedom, and energy transitions. It is asserted that in the transition region near level crossing it is not possible to go over smoothly from the adiabatic to the diabatic picture of interaction. In particular, near level crossing the mutual influence of the collective and intrinsic dynamics is especially strongly manifested in many-particle systems.

We present here an investigation of the problem of level crossing and problems associated with it on the basis of exactly solvable models in the framework of the method of adiabatic representation.

3. THE METHOD OF ADIABATIC REPRESENTATION

We shall proceed from the Schrödinger equation (3) with a time-independent Hamiltonian. The proposed approach can be used to investigate more complicated equations (1) with a time-dependent Hamiltonian after implementation of the procedure of unitary transformation (2)–(9) and reduction of Eq. (1) to Eq. (3) with a time-independent Hamiltonian (4). Here we shall also use the technique of unitary transformations of the type (4) in the coordinate space.

The adiabatic representation, in which for the Hamiltonian h one introduces the decomposition

$$h=h^{\mathrm{s}}\otimes I+h^{\mathrm{f}}, \tag{20}$$

is formulated on a Hilbert fiber bundle $\mathscr{H}=\int_{\mathsf{B}}\oplus\mathscr{F}_{\mathbf{x}}d\mu(\mathbf{x})$, where B is the base, $\mu(\mathbf{x})$ is a positive measure on B, and the fibers $\mathscr{F}_{\mathbf{x}}\ni\{\phi_n(\mathbf{x},\cdot)\}$ are formed from eigenfunctions $\phi_n(\mathbf{x},\cdot)$ of self-adjoint operators $h^{\mathrm{f}}(\mathbf{x})$ parametrized by the points $\mathbf{x}\in\mathsf{B}$:

$$h^{\mathrm{f}}(\mathbf{x})\phi_n(\mathbf{x};\cdot)=\mathscr{E}_n(\mathbf{x})\phi_n(\mathbf{x};\cdot);$$

$$h^{\mathrm{f}}(\mathbf{x})=-\Delta_{\mathbf{y}}+V(\mathbf{x},\mathbf{y}). \tag{21}$$

Since the operators $h^{\mathrm{f}}(\mathbf{x})$ act on the Hilbert fibers $\mathscr{F}_{\mathbf{x}}$, they are called the fibers of the operator H: $H=\int_B\oplus h^{\mathrm{f}}(x)dx$. The operator $h^{\mathrm{s}}\otimes I$ acts as h^{s} with respect to the slow (s) variables $\mathbf{x}$ and as the identity operator with respect to the fast (f) variables $\mathbf{y}\in\hat{\mathsf{M}}$.

In such an approach, the total wave function $\psi(\mathbf{X})$, $\mathbf{X}=\{\mathbf{x},\mathbf{y}\}\in\mathsf{M}$, of the system can be represented in the form of the expansion

$$|\psi(\mathbf{X})\rangle=|n\rangle\langle n|\psi\rangle=\sum_n \int \phi_n(\mathbf{x};\cdot)F_n(\mathbf{x}) \tag{22}$$

with respect to the eigenstates $\phi_n(\mathbf{x};\cdot)$ of the Hamiltonian of the fast motion $h^{\mathrm{f}}(\mathbf{x})$ $\forall\mathbf{x}\in B$ (21), which, since this Hamiltonian is self-adjoint, form a complete orthonormal set:

$$\langle n|m\rangle=\int \phi_n^{\dagger}(\mathbf{x};\mathbf{y})\phi_m(\mathbf{x};\mathbf{y})d\mathbf{y}=\delta_{nm};$$

$$|n\rangle\langle n|=\delta(\mathbf{y}-\mathbf{y}'),\forall\mathbf{x}. \tag{23}$$

The symbol $\Sigma\int$ in (22) denotes summation over the states of the discrete spectrum $\mathscr{E}_n(\mathbf{x})\in\sigma_d(h^{\mathrm{f}}(\mathbf{x}))$ and integration over the states of the continuous spectrum $\mathscr{E}_k(\mathbf{x})\in\sigma_c(h^{\mathrm{f}}(\mathbf{x}))$. If the functions are defined on a compact set of values $\mathbf{y}\in\hat{M}$, then they are all square-integrable, and the spectrum is purely discrete, as occurs in the case of hyperspherical parametrization of the space, when $\mathbf{y}\in S^M$ is a set of angles. In the general case, since the scattering states $\psi_k(\mathbf{x};\cdot)\equiv\psi(k,\mathbf{x};\cdot)$ together with the states of the discrete spectrum form a complete set, they must be taken into account in the expansion (1) even though they do not belong to L_2. Depending on the particular formulation of the problem, a compact or a noncompact base manifold will be used.

Substitution of the expansion (22) for ψ in the original multidimensional Schrödinger equation and averaging over the fast variables of the intrinsic motion leads to a many-channel system of equations with covariant derivative for the expansion coefficients $F=\{F_n\}$:

$$[-(\nabla\otimes I-iA(\mathbf{x}))^2+V(\mathbf{x})\otimes I-P^2]F(\mathbf{x})=0,$$
$$P=\mathrm{diag}(p_n), \tag{24}$$

where the operators $A(\mathbf{x})$ and $V(\mathbf{x})$ act as operator-valued entities for the matrix components of the effective vector and scalar potentials generated by the basis functions:

$$A_{nm}(\mathbf{x})=\langle\phi_n(\mathbf{x};\cdot)|i\nabla_{\mathbf{x}}|\phi_m(\mathbf{x};\cdot)\rangle. \tag{25}$$

This definition of the matrix elements of A, which realize the exchange interaction, is the same as the definition (17), except that A occurs in the different equations (15) and (24). In both cases, the matrix elements of A play the role of coupling functions and are sources of topological effects. The potential matrix in (24) is diagonal ($V(\mathbf{x})=\mathrm{diag}\{\mathscr{E}(\mathbf{x})\}$), and its elements are equal to the energy levels $\mathscr{E}_m(\mathbf{x})$ of the instantaneous Hamiltonian $h^{\mathrm{f}}(\mathbf{x})$ (21):

$$V_{nm}(\mathbf{x})=\langle\phi_n(\mathbf{x};\cdot)|h^{\mathrm{f}}(\mathbf{x})|\phi_m(\mathbf{x};\cdot)\rangle=\mathscr{E}_n(\mathbf{x})\delta_{nm}. \tag{26}$$

In the general case, $V(\mathbf{x})$ must be augmented by a certain potential matrix $V^{\mathrm{s}}(\mathbf{x})$, which is contained only in the system of equations (24) $V^{\mathrm{s}}_{nm}(\mathbf{x})=\langle\phi_n(\mathbf{x};\cdot)|V^{\mathrm{s}}(\mathbf{x},\cdot)|\phi_m(\mathbf{x};\cdot)\rangle$. Note that the representation (22) of the total wave function ψ must be invariant with respect to the choice of the basis set of functions

$$|\psi\rangle=|n\rangle\langle n|\psi\rangle=|n\rangle\mathscr{U}^{\dagger}\mathscr{U}\langle n|\psi\rangle. \tag{22a}$$

Then, as follows from (22), $|\phi(\mathbf{x};\cdot)\rangle$ is replaced by $|\phi'(\mathbf{x};\cdot)\rangle$ and $F(\mathbf{x})$ by $F'(\mathbf{x})$:

$$|\phi'(\mathbf{x};\mathbf{y})\rangle=|\phi(\mathbf{x};\mathbf{y})\rangle\mathscr{U}^{\dagger}(\mathbf{x}),\quad F'(\mathbf{x})=\mathscr{U}(\mathbf{x})F(\mathbf{x}). \tag{27}$$

It is easy to show that as a result of such a transformation the effective matrices $V(\mathbf{x})$ and $A(\mathbf{x})$ transform as gauge scalar and vector potentials:

$$V'(\mathbf{x})=\mathscr{U}(\mathbf{x})V(\mathbf{x})\mathscr{U}^{-1}(\mathbf{x}),$$
$$A'_{\nu}(\mathbf{x})=\mathscr{U}A\mathscr{U}^{-1}-i\mathscr{U}^{-1}\partial_{\nu}\mathscr{U}, \tag{28}$$

and the extended derivative $D_{\nu}\equiv\partial_{\nu}\otimes I-iA_{\nu}(\mathbf{x})$ in (24) transforms as a covariant derivative. Here $\mathscr{U}(\mathbf{x})$ is the gauge transformation. The unitarity of $\mathscr{U}$ follows from the condition of completeness of the sets $\{\phi\}$ and $\{\phi'\}$ in each fiber $\mathscr{F}_{\mathbf{x}}$:

$$\mathscr{U}(\mathbf{x})\mathscr{U}^{-1}(\mathbf{x})=1,\ \mathscr{U}^{\dagger}(\mathbf{x})=\mathscr{U}^{-1}(\mathbf{x}),\ \forall\mathbf{x}\in B. \tag{29}$$

Using the orthogonality and completeness relations (23), we define a frame $|e(\mathbf{y})\rangle\equiv|\phi(\mathbf{x}_0;\mathbf{y})\rangle$ at some fixed point $\mathbf{x}=\mathbf{x}_0$. The moving frame $|\phi(\mathbf{x};\cdot)\rangle$ is related to the fixed frame $|e(\cdot)\rangle$ by means of a unitary bilocal operator $\mathscr{U}(\mathbf{x})\equiv\mathscr{U}(\mathbf{x},\mathbf{x}_0)$:

$$|\phi(\mathbf{x};\cdot)\rangle=|e(\cdot)\rangle\mathscr{U}(\mathbf{x},\mathbf{x}_0),\ \mathscr{U}(\mathbf{x},\mathbf{x}_0)=\langle e(\cdot)|\phi(\mathbf{x};\cdot)\rangle, \tag{30}$$

which realizes parallel transport of the frame from $\mathbf{x}_0$ to $\mathbf{x}$. Using now the unitary gauge transformation

$$\mathscr{U}(\mathbf{x},\mathbf{x}_0)=\exp i\int_{\mathbf{x}_0}^{\mathbf{x}}A(\mathbf{x}')d\mathbf{x}',$$

we go over to the representation of the fixed basis $|e(\mathbf{x}_0,\mathbf{y})\rangle$. Under the condition of a pure gauge, A is annihilated, and the system of equations (24) reduces to an ordinary many-channel system of equations with potential decoupling for the new coefficients F', which are related to the old F by the relation (27):

$$[-(1/2)\Delta+\mathscr{U}(\mathbf{x})V(\mathbf{x})\mathscr{U}^{-1}(\mathbf{x})-P^2]F'(\mathbf{x},P)=0. \tag{31}$$

After this, it is possible to apply to the system of equations (31) the standard methods of the many-channel inverse problem, provided that the corresponding scattering matrix is known together with the information about the states of the discrete spectrum: their positions and normalizations. We note that this is possible if $A(\mathbf{x})$ has no singularities.

3.1. Inverse scattering problem in the adiabatic representation

Thus, the inverse problem in the adiabatic approach reduces to a mutually consistent solution of the many-channel inverse problem for a system of equations of the gauge type (24) and the parametric problem (21) in a fiber. On the basis of the many-channel and single-channel technique of Bargmann potentials, a method of analytic modeling of the effective interactions and corresponding solutions in composite quantum systems with several degrees of freedom was formulated in Refs. 5, 22, and 23. For potentials of Bargmann type, the kernels of the equations of the inverse problem can be represented as a sum of factorized terms; as a result, the

integral equations of the inverse problem reduce to systems of algebraic type and can be solved in closed form.

In this paper, we demonstrate the possibilities of the method for the examples of two-dimensional exactly solvable models, $\mathbf{X}=\{x,y\}$. It is possible to recover the potential $V(\mathbf{x})$ and find the functions of the moving frame by means of the formalism of the inverse problem for the "fast" equation (21) in the case of a parametric dependence of the scattering data on the slow variables x, this dependence being determined, in its turn, by the solution of the inverse problem for the "slow" system of equations (24). For a parameter dependence of the spectral data $\{S(x,k),\mathscr{E}_i(x),\gamma_i^2(x)\}$ that is given in advance, one initially recovers the two-dimensional potential $V(x,y)$ by solving the parametric inverse problem and determines the basis functions $\psi_i(x;y)$, from which the matrix of the induced vector potential $A(x)$ can be found from (25). After this, it is possible to solve the system of equations (24) for $F(x)$ and the S matrix or the system of equations (16) for the amplitudes $c(t)$. In contrast to the well-known models—Landau–Zener,[20,21] resonant charge exchange,[28] Rozen–Zener–Demkov[26,27]—in the given formulation there is no stringent restriction on the form of the exchange interaction $A(x)$ and the law of variation of the interlevel separation $\Delta(x)=\mathscr{E}_n(x)-\mathscr{E}_m(x)$ when one is obtaining the transition amplitudes in either the classical approach (Refs. 20, 21, and 26–28) or the quantum-mechanical approach.[30–32]

3.2. Exactly solvable models for the parametric problem

By analogy with the ordinary technique of Bargmann potentials, the Jost functions of the parametric equations (14) and (21) are chosen in rational form.[5,22] However, they now depend on the external coordinate variable through the dependence on it of the spectral parameters $\alpha(x)$ and $\beta(x)$:

$$f(x;k)=\mathring{f}(k)\prod\frac{k-i\alpha(x)}{k+i\beta(x)}. \tag{32}$$

The parametric Jost function (32) has N simple poles on the trajectories of the momenta in the parameter space $B\ni x$ at $k=i\beta_j(x)$ and N simple zeros at $k=i\alpha_j(x)$ and not at separate points as in the ordinary inverse problem. Moreover, $\alpha(x)$ contains not only zeros on the imaginary half-axis, corresponding to bound states $\operatorname{Re}\varkappa_j(x)=0$, $\operatorname{Im}\varkappa_j(x)>0$ but also zeros in the lower half-plane with $\operatorname{Im}\nu_j(x)<0$ (the number of simple poles β_j is equal to the number of $\varkappa_j$ and ν_j values taken together). Then the S matrix and the spectral function take the form

$$s(x;k)=\mathring{s}(k)\prod\frac{(k+i\alpha(x))(k+i\beta(x))}{(k-i\beta(x))(k-i\alpha(x))}, \tag{33}$$

$$\rho(x;k)=\mathring{\rho}(k)\prod\frac{(k-i\beta(x))(k+i\beta(x))}{(k+i\alpha(x))(k-i\alpha(x))}. \tag{34}$$

The functions $\mathring{s}(k)$ and $\mathring{\rho}(k)$ are known, since the potential $\mathring{V}(y)$ is known; in a special case, $\mathring{V}(y)=0$. For $s(x,k)$ and $\rho(x,k)$ such as (33) and (34), the kernels $Q(x;y,y')$ and $K(x;y,y')$ of the integral equations of the inverse problem

$$K(x;y,y')+Q(x;y,y')+\int_{y(0)}^{\infty(y)}K(x;y',y'')Q(x;y',y'')dy''=0 \tag{35}$$

become degenerate and can be represented as a sum of several terms with a factorized coordinate dependence with respect to the fast variable y:

$$Q(x;y,y')=\sum_n^m\gamma_n^2(x)\mathring{\phi}(i\varkappa_n(x),y)\mathring{\phi}(i\varkappa_n(x),y')-\sum_n^{\dot m}\mathring{\gamma}_n^2(x)\mathring{\phi}(i\mathring{\varkappa}_n(x),y)\mathring{\phi}(i\mathring{\varkappa}_n(x),y'), \tag{36}$$

$$K(x;y,y')=-\left\{\sum_n^m\gamma_n^2(x)\phi(i\varkappa_n(x),y)\mathring{\phi}(i\varkappa_n(x),y')-\sum_n^{\dot m}\mathring{\gamma}_n^2(x)\phi(i\mathring{\varkappa}_n(x),y)\mathring{\phi}(i\mathring{\varkappa}_n(x),y')\right\}. \tag{37}$$

As a consequence of this, the integral equations (35) become algebraic equations, and the spherically nonsymmetric potential and the solutions corresponding to it can be expressed in closed analytic form in terms of the known solutions of the nonparametric problem with potential $\mathring{V}(y)$ and in terms of the spectral characteristics of two problems: the nonparametric problem with the potential $\mathring{V}(y)$ and the parametric problem (14), (21) with potential $\mathring{V}(y)+V(x,y)$:

$$V(x;y)=\mathring{V}(y)\mp2\frac{d}{dy}K(x;y,y), \tag{38}$$

$$\phi(x;k,y)=\mathring{\phi}(k,y)+\int_{y(0)}^{\infty(y)}K(x;y,y')\mathring{\phi}(k,y')dy'. \tag{39}$$

Here $\phi(x;k,y)$ are the solutions of (21) or (14) with the required potential $V(x;y)$ for arbitrary k represented in terms of the solutions $\phi[i\varkappa_n(x),y]$ of the same equation for $\mathscr{E}_n(x)=-\varkappa_n^2(x)$:

$$\phi[i\varkappa_n(x),y]=\sum_j^m\mathring{\phi}[i\varkappa_j(x),y]P_{jn}^{-1}(x;y), \tag{40}$$

where

$$P_{nj}(x;y)=\delta_{nj}+\gamma_n^2\int_{y(0)}^{\infty(y)}\mathring{\phi}[i\varkappa_n(x),y']\mathring{\phi}[i\varkappa_j(x),y']dy'.$$

By $\mathring{\phi}(k,y)$ it is necessary to understand the Jost solutions in Marchenko's approach or the regular solutions in the Gel'fand–Levitan approach corresponding to the potential $\mathring{V}(y)$. The limits of integration and the signs in (39) and (40) depend on the specific approach within the inverse scattering problem. The limits from y to ∞ and the minus sign correspond to Marchenko's formulation. The limits $[0,y]$ and the plus sign correspond to the Gel'fand–Levitan approach.

We consider the problem of constructing transparent symmetric potentials with respect to the fast variable y and solutions corresponding to them for $\overset{\circ}{V}(y)=0$, following the studies of Refs. 24 and 33. The relations for the potentials and the solutions (38)–(40) can be expressed in terms of the normalized eigenfunctions (14):

$$\psi(x;y)=\gamma_n(x)f_+(x;k,y), \tag{41}$$

where $f_+(x;k,y)$ are the Jost solutions of Eq. (14). For transparent potentials, $\mathscr{S}(x;k)=0$, $\forall x$, and in the kernel $Q(x;y,y')$, which is determined by the scattering data, there remains only a sum over the bound states:

$$Q(x;y,y')=\sum_n \gamma^2 \exp(-\varkappa_n(x)y)\exp(-\varkappa_n(x)y'). \tag{42}$$

If the transparent potentials are symmetric with respect to the fast variable y, then the potential and the solutions are completely determined by the values $\mathscr{E}_n(x)=-\varkappa_n^2(x)$ of the energy levels, since the corresponding normalizations are determined by the energies of the bound states:

$$\gamma_n^2(x)=i\mathrm{Res}\mathscr{S}^r(k)_{/k=i\varkappa_n(x)}$$

$$=2\varkappa_n(x)\prod_{m\neq n}\left|\frac{\varkappa_m(x)+\varkappa_n(x)}{\varkappa_m(x)-\varkappa_n(x)}\right|. \tag{43}$$

We rewrite the kernel (42) in the form

$$Q(x;y,y')=-\sum_n^m \lambda_n(x;y)\lambda_n(x;y'), \tag{44}$$

introducing the functions

$$\lambda_n(x;y)=\gamma_n(x)\exp(-\varkappa_n(x)y).$$

After this, the kernel $K(x;y,y')$ of the generalized shift can be represented as follows:

$$K(x;y,y')=-\sum_n^m \gamma_n(x)\psi_n(x;y)\exp(-\varkappa_n(x)y')$$

$$=-\sum_n^m \psi_n(x;y)\lambda_n(x;y). \tag{45}$$

For the normalized eigenfunctions, we obtain

$$\psi_n(x;y)=\sum_j^m \lambda_j(x;y)C_{jn}^{-1}(x;y) \tag{46}$$

with matrix of coefficients

$$C_{jn}(x;y)=\delta_{jn}+\frac{\lambda_j(x;y)\lambda_n(x;y)}{\varkappa_n(x)+\varkappa_j(x)}. \tag{47}$$

Finally, the kernel $K(x;y,y')$ and the potential can be represented in the form

$$K(x;y,y')=-\sum_n^m\sum_j^m \lambda_j(x;y)C_{jn}^{-1}(x;y)\lambda_n(x;y),$$

$$V(x;y)=-4\sum_n^m \varkappa_n(x)\psi_n^2(x;y). \tag{48}$$

We note once more that these relations are obtained for the specific case of a vanishing reflection coefficient: $\mathscr{S}(x;k)=0$, $\forall x$.

We can now determine the matrix elements of the operator of the connection $A(x)$ in both the approaches (25) and (17) that are induced by the functions (46) of the parametric fast problem. We mention here the decisive role of the connection $A(x)$ in the occurrence of phenomena such as level crossing, geometric phases $\oint_c A(x)\cdot dx$, and the Aharonov–Bohm effect, which can also be interpreted as a special case of the Aharonov–Anandan phase.

4. TWO-DIMENSIONAL EXACTLY SOLVABLE MODELS

We give below some specific examples of the construction of two-dimensional potentials, the analytic solutions of the parametric problem that correspond to them, and the matrix elements of the exchange interaction for a given functional dependence of the terms.

4.1. Construction of exactly solvable models in Marchenko's approach

We give an example of a simple case of transparent symmetric potentials with one term. Let $\overset{\circ}{V}(y)=0$. The relations (41)–(48) give the generalized Eckart potential

$$V(x;y)=-2\frac{2\varkappa(x)\gamma^2(x)\exp(-2\varkappa(x)y)}{[1+(\gamma^2(x)/2\varkappa(x))\exp(-2\varkappa(x)y)]^2}, \tag{49}$$

which can be transformed to a simpler form well known in the theory of solitons,

$$V(x;y)=-\frac{2\varkappa^2(x)}{\cosh^2[\varkappa(x)(y-y_0(x))]}, \tag{50}$$

by using the substitution

$$\exp(2\varkappa(x)y_0)=\gamma^2(x)/2\varkappa(x) \tag{51}$$

and the transformation

$$\{1+\exp[-2\varkappa(x)(y-y_0(x))]\}^2=4\cosh^2[\varkappa(x)(y-y_0(x))]\exp[-2\varkappa(x)(y-y_0(x))].$$

The Jost solutions that correspond to it on the trajectory $k=\varkappa(x)$ and also for arbitrary values of k can be written in the explicit form

$$f(i\varkappa(x),y)=\frac{\exp(-\varkappa(x)y)}{1+\exp[-2\varkappa(x)(y-y_0(x))]}, \tag{52}$$

$$f_\pm(x;k,y)=\exp(\pm iky)$$

$$\times\left\{1-\frac{\exp[-2\varkappa(x)(y-y_0(x))]}{1+\exp[-2\varkappa(x)(y-y_0(x))](\varkappa(x)\mp ik)}\right\}. \tag{53}$$

We give the explicit expression for the normalized wave function $\psi(x,y)$ of the potential (50) corresponding to the term $\mathscr{E}(x)=-\varkappa^2(x)$:

$$\psi(x,y)=\frac{\sqrt{\varkappa(x)/2}}{\cosh[\varkappa(x)(y-y_0(x))]}. \tag{54}$$

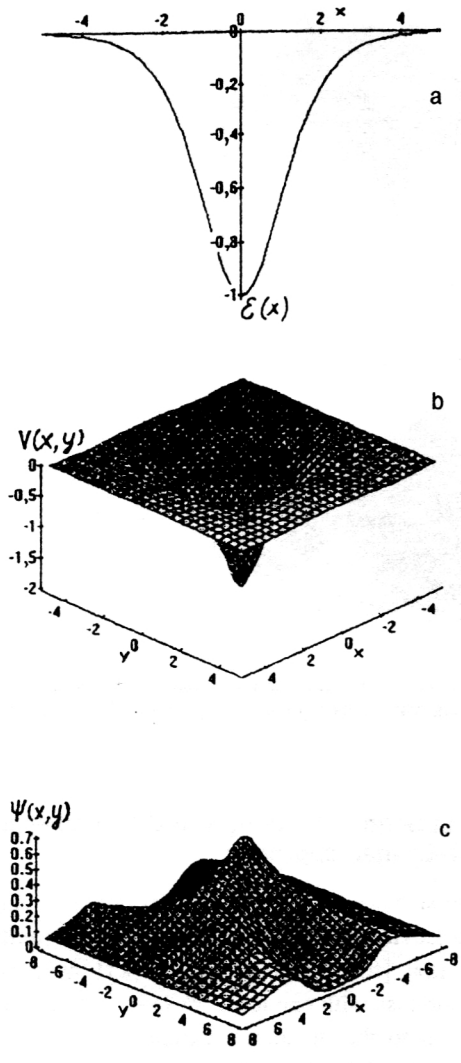

FIG. 1. Transparent two-dimensional potential $V(x;y)$ (b) symmetric with respect to both coordinates x and y and with one bound state. a) The term $\mathscr{E}(x)$ corresponds to a potential that is transparent and symmetric with respect to x and has a bound state $E=-0.5$. c) Eigenfunction $\psi(x;y)$ of the parametric problem corresponding to the term (a).

It is easy to show that the potential is related to the normalized wave function by

$$V(x;y)=-4\varkappa(x)\psi^2(x;y). \tag{55}$$

Note that in the expressions (50) and (52)–(55) the function $y_0(x)$ is always related to the normalization function $\gamma^2(x)$ by Eq. (51). If the normalization function is chosen in accordance with (43),

$$\gamma^2(x)=2\varkappa(x),$$

i.e., if it corresponds to a symmetric potential transparent with respect to y, then $y_0(x)=0$, $\forall x$. This is not remarkable, since, as can be seen from the expression (50), for $y_0(x)=0$ the potential is symmetric with respect to $y=0$. If, in turn, the behavior of the term $\mathscr{E}(x)$ (Fig. 1a) corresponds to the potential $V(x)\equiv\mathscr{E}(x)$ of the "slow" equation,

$$\mathscr{E}(x)=-\frac{2\varkappa^2}{\cosh^2(\varkappa(x-x_0))}, \tag{56}$$

which is symmetric and transparent with respect to x, $y^2=2\varkappa$, $x_0=0$, then the relations (50) or (55), (54), and (53) give a two-dimensional potential that is transparent and symmetric with respect to both variables and has one bound state $E=-\varkappa^2$ (Fig. 1b), and they also give the two-dimensional wave functions of the bound state, $\psi(x;y)$ (Fig. 1c), and of the continuous spectrum $f_{\pm}(x;k,y)$ of the parametric problem.

We now give some examples of a two-dimensional potential and of the analytic solutions of the parametric problem with two terms that correspond to it. Depending on the particular problem, the terms can, in general, behave in different ways. In the presented case, we specify them as follows:

$$\varkappa_n(x)=\frac{b_n}{\cosh(a_n x)}+c_n, \quad n=1,2,$$

where a_n, b_n, c_n are certain constants that determine the form of the terms $\mathscr{E}_n(x)=-\varkappa_n^2(x)$. We choose the normalization in the form (43), which corresponds to the case of transparent symmetric potentials. Then from the relations (37)÷(38) or (46)–(48) we obtain the two-dimensional potential (Figs. 2b, 3b, and 4b) and the two-dimensional wave functions (Figs. 2c and 2d, 3c and 3d, and 4c and 4d) that correspond to the chosen terms (Figs. 2a and 3a). As can be seen from comparison of Figs. 2b, 3b, and 4b with Figs. 2c, 2d, 3c, 3d, 4c, and 4d, thehavior of the potential and wave functions as the levels approach each other becomes more complicated.

The induced connection (Fig. 5) calculated in accordance with (25) and (46) is singlular. The two crossing terms in Fig. 5b are defined as follows: $\mathscr{E}_1(x)=-2/\cosh^2(x/2)$, $\mathscr{E}_2(x)=-1/\cosh^2(x/3)$. For comparison, Fig. 5a shows the matrix element $A_{12}(x)$ for the nonintersecting terms: $\mathscr{E}_1(x)=-2/\cosh^2(x/2)$, $\mathscr{E}_2(x)=-1/(\cosh(x/3)+0.5)^2$. Berry[8] also noted the special role of degeneracy of the states in the parameter space; the degeneracy is important for the existence of a nontrivial topological phase. A separate paper (see also Ref. 15 and the final section of this paper) will be devoted to investigation of the situation with crossing and quasicrossing of levels.

We give an example of a two-dimensional exactly solvable model corresponding to the situation with three terms. Figure 6 demonstrates a two-dimensional potential that is transparent with respect to y and the basis wave functions that correspond to it.

For completeness of the picture, we also give an example of periodic variation of terms (Figs. 7–9). As can be seen from comparison of Figs. 7b, 8b, and 9b, approach of the levels to each other influences the structure of the potential that arises with respect to y. The behavior of the central layer is not sensitive to the approach of the levels to each other, whereas the role of the second and third layers increases and becomes comparable with the role of the main layer. In particular, the third layer has increased, and a fourth, previously absent, has arisen. The potential is symmetric with respect to y and also, of course, with respect to

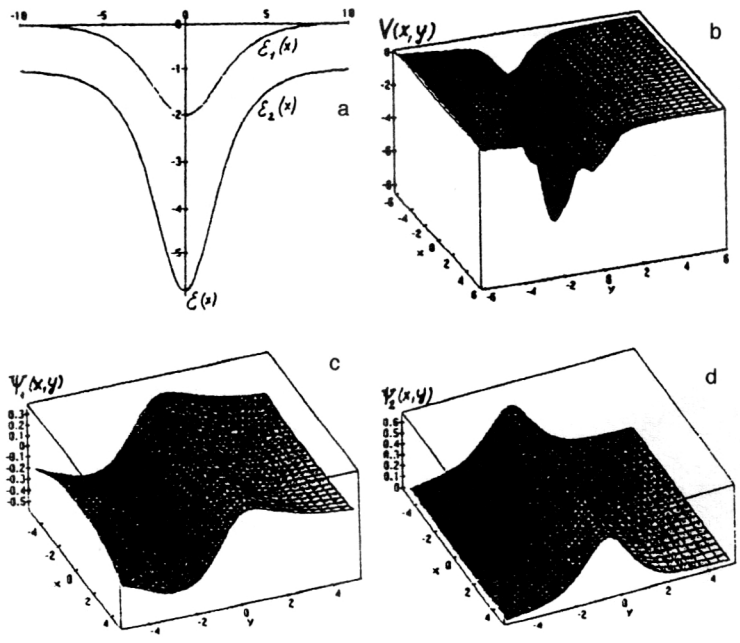

FIG. 2. Potential $V(x;y)$ transparent with respect to the variable y (b) symmetric with respect to both coordinates x and y, and with two terms symmetric with respect to x (a) $\mathscr{E}_n(x)=-\varkappa_n^2(x)$: $\varkappa_1(x)=2^{1/2}/\cosh(x/3)$, $\varkappa_2(x)=2^{1/2}/\cosh(x/2)+1$; c), d) eigenfunctions $\psi_1(x;y)$ and $\psi_2(x;y)$ of the parametric problem with the normalizations (43).

x. Note that the addition of several oscillations in the terms has hardly any influence on the behavior of the potential with respect to the fast variable y. The potential and also the wave functions of the eigenstates of the parametric problem have a clear periodic structure with respect to the slow variable x, as was to be expected from the periodic behavior of the terms. The main layer has the maximum values corresponding to the greatest difference between the terms, whereas the following layers give an increasing contribution near the minimum separation between the terms.

4.2. Construction of exactly solvable models in the Gel'fand–Levitan approach

We first consider the simplest example possible in the Gel'fand–Levitan approach. It is that of a spectrally equivalent potential [for $E>0$, $\rho(x;k)=\mathring{\rho}(k)$] with one bound state, in our case with one term, $\mathscr{E}(x)=-\varkappa^2(x)$. To simplify the problem to the maximum extent, we again choose the reference potential $\mathring{V}(y)\equiv 0$. Then the corresponding Jost function can be written in the form

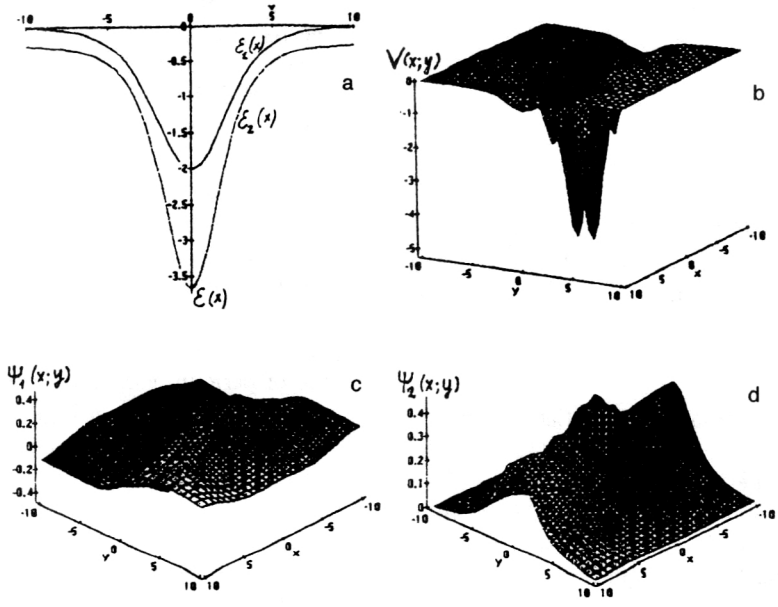

FIG. 3. The potential $V(x;y)$ (b) and the functions $\psi_1(x;y)$ and $\psi_2(x;y)$ (c, d) corresponding, as in Fig. 2, to a situation with two terms (a) $\varkappa_1(x)=2^{1/2}/\cosh(x/3)$, $\varkappa_2(x)=2^{1/2}/\cosh(x/2)+0.5$ that, however, have a smaller separation.

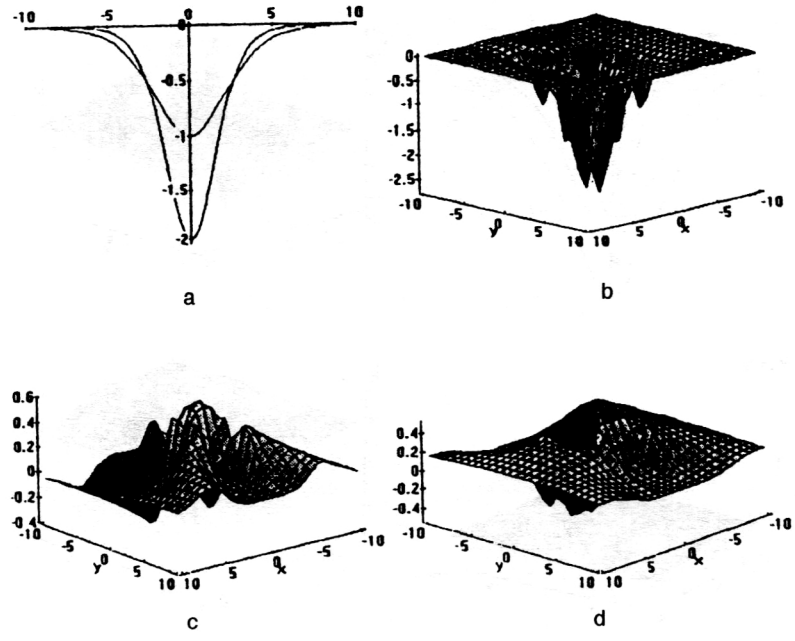

FIG. 4. The potential $V(x;y)$ (b) and the functions $\psi_1(x;y)$ and $\psi_2(x;y)$ (c, d) corresponding to a situation with two terms (a) $\varkappa_1(x)=2^{1/2}/\cosh(x/2)$, $\varkappa_2(x)=2^{1/2}/\cosh(x/3)$ that have one common point.

$$f(x;k)=\frac{k-i\varkappa(x)}{k+i\varkappa(x)}. \tag{57}$$

Hence, as follows from (34), $\rho(x,k)=2k/\pi=\mathring{\rho}(k)$, i.e., the spectral function of the continuous spectrum is identical to the spectral function of free motion. Then from the relations (35) and (37) we obtain the expressions for the kernels of the fundamental Gel'fand–Levitan equations:

$$Q^{\mathrm{GL}}(x;y,y')=c^2(x)\,\frac{\sinh[\varkappa(x)y]\sinh[\varkappa(x)y']}{\varkappa^2(x)}; \tag{58}$$

$$K^{\mathrm{GL}}(x;y,y')=-c^2(x)\phi(i\varkappa(x),y)\,\frac{\sinh[\varkappa(x)y'])}{\varkappa(x)}, \tag{59}$$

and for the regular solutions on the trajectory $k=\varkappa(x)$ we obtain

$$\phi(i\varkappa(x),y)=\frac{\varkappa(x)\sinh(\varkappa(x)y)}{\varkappa^2(x)+1/2c^2(\sinh(2\varkappa(x)y)/2\varkappa(x))-y)}. \tag{60}$$

Finally, taking into account (59) and (60) and substituting the result in the relations (38) and (39), we obtain explicit relations for the two-dimensional potential $V(x;y)$ and the solutions corresponding to it for arbitrary k with parametric dependence on x:

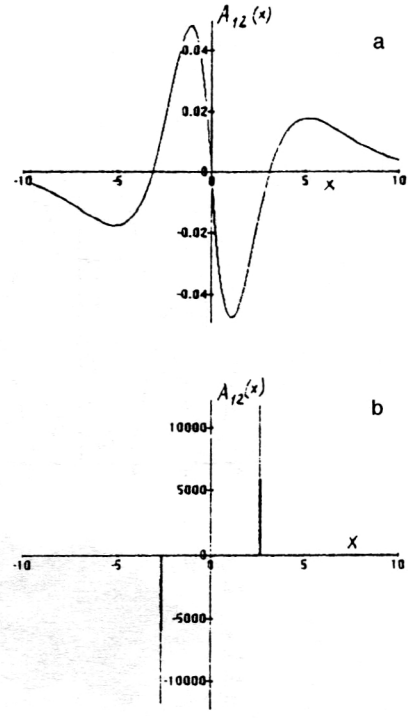

FIG. 5. a) Matrix element $A_{12}(x)$ for terms without crossing: $\mathscr{E}_1(x)=-2/\cosh^2(x/2)$, $\mathscr{E}_2(x)=-(1/\cosh(x/3+0.5)^2$; b) matrix element $A_{12}(x)$ for terms having two degenerate points: $\mathscr{E}_1(x)=-2/\cosh^2(x/2)$, $\mathscr{E}_2(x)=-1/(\cosh^2(x/3)$.

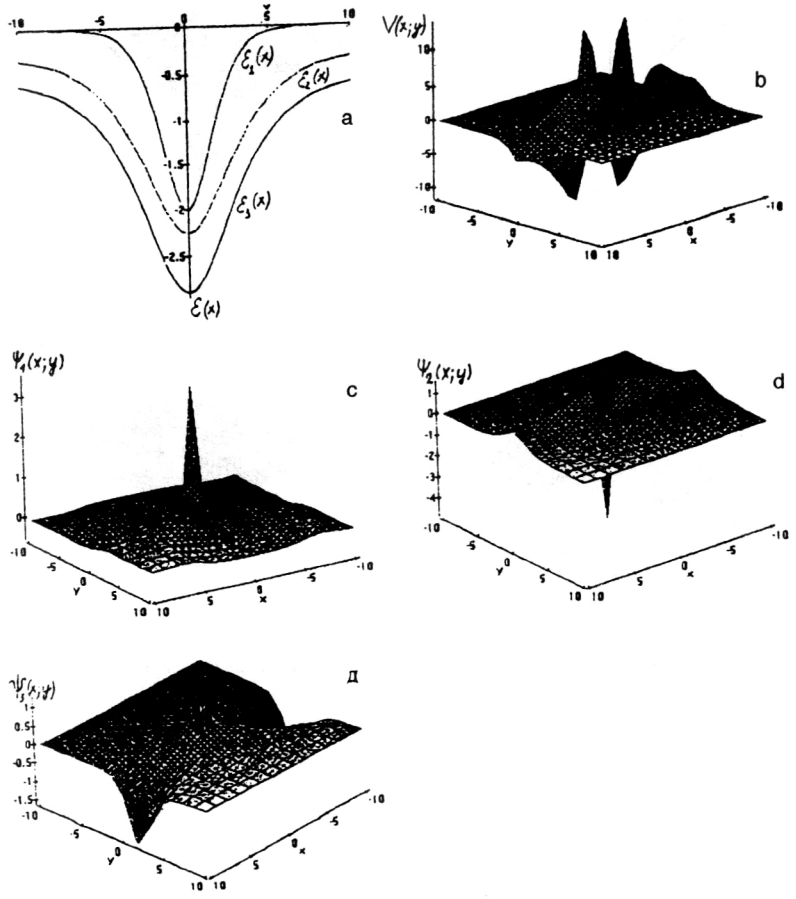

FIG. 6. Potential $V(x;y)$ (b) that is transparent with respect to the variable y and has three bound states for the terms (a) $\mathscr{E}_n(x) = -\varkappa_n^2(x)$: $\varkappa_1(x) = 2^{1/2}/\cosh(x/2)$, $\varkappa_2(x) = 1/\cosh(x/2) + 0.5$, $\varkappa_3(x) = 1/\cosh(x/3) + 0.7$; c), d), e) the eigenfunctions $\psi_1(x;y)$, $\psi_2(x;y)$, and $\psi_3(x;y)$, respectively, of the parametric problem with the normalizations (43).

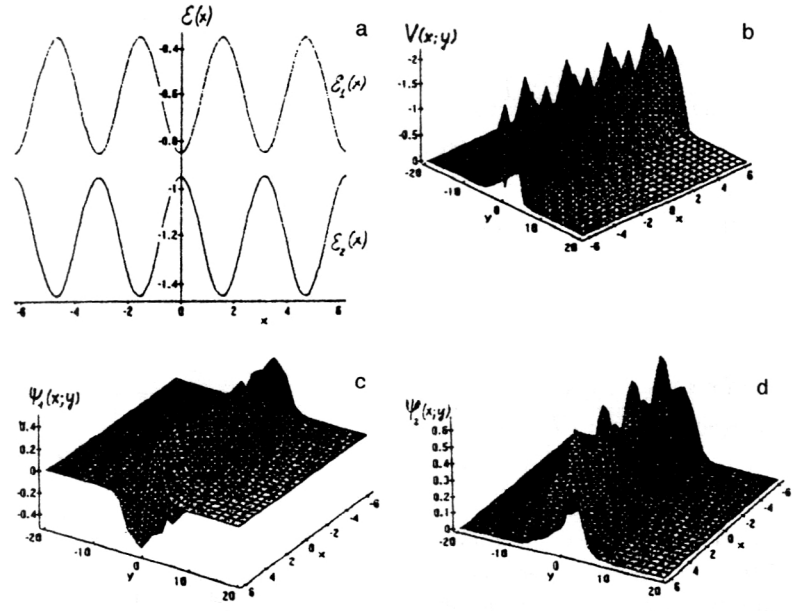

FIG. 7. Potential $V(x;y)$ (b) symmetric and transparent with respect to the fast variable y and the corresponding eigenfunctions (c, d) $\psi_1(x;y)$ and $\psi_2(x;y)$ for terms (a) that depend periodically on x: $\mathscr{E}_1(x) = -(1/4)\cos(2x) - 0.6$, $\mathscr{E}_2(x) = (1/4)\cos(2x) - 1.2$.

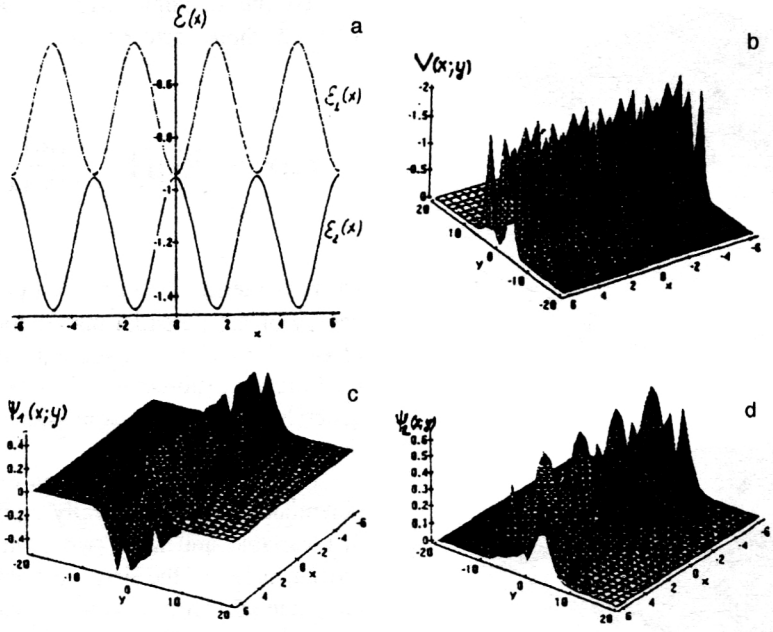

FIG. 8. Potential $V(x;y)$ (b) symmetric and transparent with respect to the fast variable y and the corresponding eigenfunctions (c, d) $\psi_1(x;y)$ and $\psi_2(x;y)$ for terms (a) that depend periodically on x: $\mathscr{E}_1(x) = -(1/4)\cos(2x) - 0.69$, $\mathscr{E}_2(x) = (1/4)\cos(2x) - 1.2$. Here the terms (a) are much closer to each other than in Fig. 7a.

$$V(x;y) = \frac{2\varkappa(x)[y/2 - c^{-2}(x)\varkappa^2(x)]\sinh 2(\varkappa(x)y) - 2\sinh^2(\varkappa(x)y)}{[c^{-2}(x)\varkappa^2(x) + 1/2[\sinh(2\varkappa(x)y)/2\varkappa(x)] - y]^2}, \tag{61}$$

$$\phi(x;k,y) = \frac{\sin ky}{k} - \frac{c^2(x)\sinh(\varkappa(x)y)[\varkappa(x)\cosh(\varkappa(x)y)\sin ky - k\sinh(\varkappa(x)y)\cos ky]}{k[\varkappa^2(x) + k^2][\varkappa^2(x) + (1/2)c^2(x)(\sinh(2\varkappa(x)y)/2\varkappa(x)) - y]}. \tag{62}$$

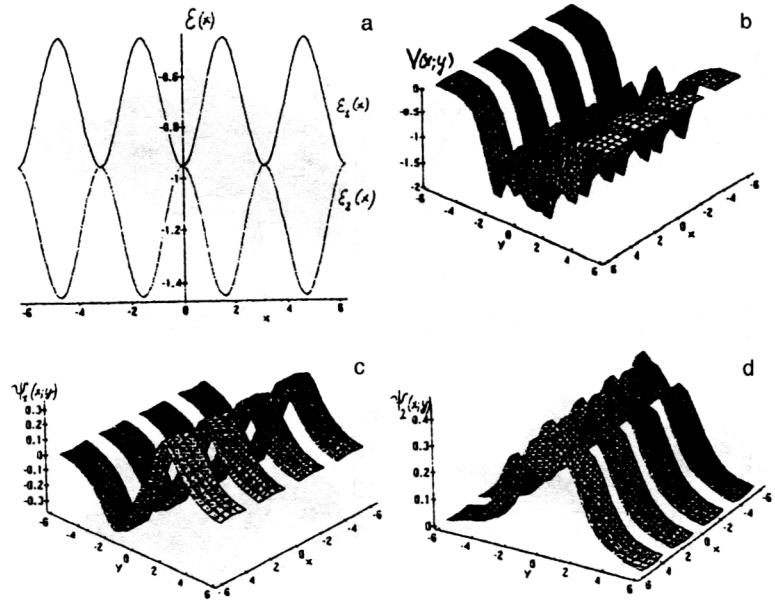

FIG. 9. Potential $V(x;y)$ (b) symmetric and transparent with respect to the variable y and the corresponding eigenfunctions $\psi_1(x;y)$ and $\psi_2(x;y)$ (c, d) for terms (a) that depend periodically on x: $\mathscr{E}_1(x) = -(1/4)\cos(2x) - 0.7$, $\mathscr{E}_2(x) = (1/4)\cos(2x) - 1.2$, and have several points of degeneracy.

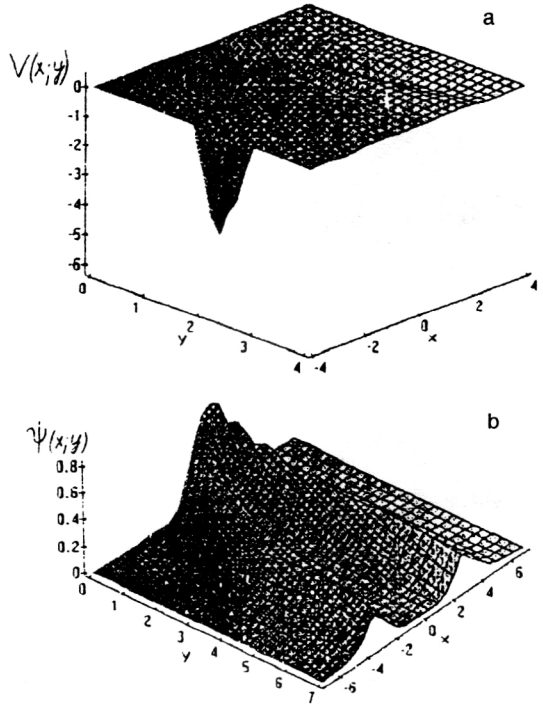

FIG. 10. Potential $V(x;y)$ (a) and eigenfunction $\psi(x;y)$ (b) of the term of Fig. 1a, calculated in the Gel'fand–Levitan approach.

Specifying explicitly the dependence of the spectral characteristics on the parameter, we model the corresponding potential and solutions of the parametric problem. Figure 10 is an example of a two-dimensional potential with one term corresponding to the potential (56), which is symmetric and transparent with respect to x, and with the corresponding wave function. Note that in this case the potential is not symmetric with respect to the other variable y.

To the example with m bound states and potential $\mathring{V}(y)\neq 0$ there corresponds

$$f_+(x;k)=\mathring{f}(k)\prod_n^m \frac{k-i\varkappa_n(x)}{k+i\varkappa_n(x)}, \tag{63}$$

where the terms $\mathscr{E}_n(x)=-\varkappa_n^2(x)$ are determined through recovery of the potential matrix for the many-channel system of equations (31) and subsequent diagonalization of it (see the following subsection). It is easy to write down an m-term generalization of the expressions (58)–(61), using the relations (35)–(40).

Figure 11 demonstrates an example of two-dimensional potentials that are spectrally equivalent with respect to the fast variable and have two terms. As in Marchenko's approach, it is not the first main layer closest to the origin in the potential that is sensitive to the behavior of the terms but rather the crests of the potential more distant with respect to y. The investigations are readily made in spherical coordinates, the angle being chosen as the fast variable, the coordinate as the slow variable, and vice versa.

In the case of transparent symmetric potentials, knowledge of the terms is sufficient for unique recovery of the two-dimensional potential and of the solutions corresponding to it, since, as can be seen from the relation (43), the normalizations are determined by the values of the energy levels. In the usual case, depending on the choice of the normalizations, a family of phase-equivalent potentials is recovered. In our case, the normalizations are in addition functions of the coordinate variable x. At the same time, the normalization functions of the Jost solutions and of the regular solutions are related to the corresponding normalizations of the fixed-basis problem by the equations

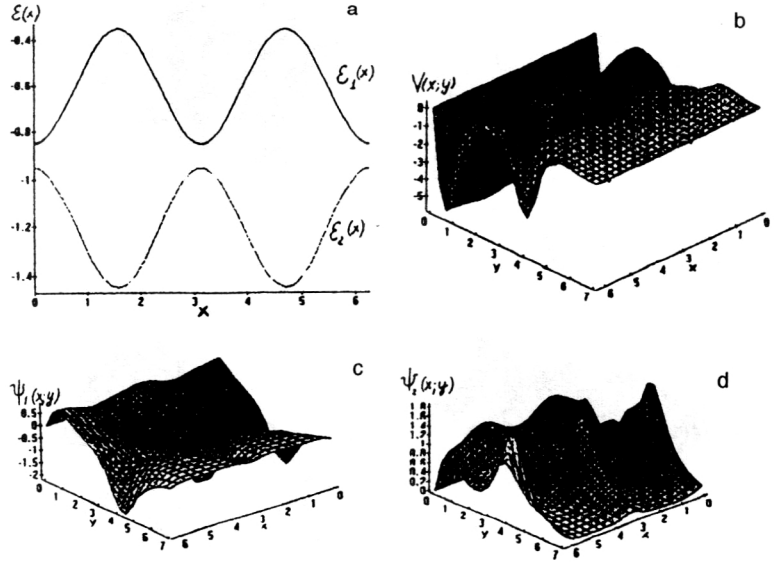

FIG. 11. Potential $V(x;y)$ (b) and eigenfunctions $\psi_1(x;y)$, $\psi_2(x;y)$ (c, d) in the Gel'fand–Levitan approach that correspond to periodic variation of the terms (a): $\mathscr{E}_1(x)=-(1/4)\cos(2x)-0.6$, $\mathscr{E}_2(x)=(1/4)\cos(2x)-1.2$.

$$\gamma_n^{-2}(x)=\sum_j^m \mathscr{U}_{nj}(x)\int_0^\infty |e_j^M(y)|^2 dy\,\mathscr{U}_{jn}(x)$$
$$=\sum_j^m \mathscr{U}_{nj}(x)M_j^{-2}\mathscr{U}_{jn}(x),$$
$$c_n^{-2}(x)=\sum_j^m \mathscr{U}_{nj}(x)N_j^{-2}\mathscr{U}_{jn}(x), \tag{64}$$
$$N_j^{-2}=\int_0^\infty |e_j^{GL}(y)|^2 dy.$$

However, the determination of the matrix of the unitary transformation $\mathscr{U}(x)$ is not a unique procedure; the determination of the potential is also not unique. Thus, nonuniqueness of the two-dimensional inverse problem has arisen. In the two-channel case, the indeterminacy associated with the diagonalization procedure disappears. This case is therefore very convenient for investigation and is used here to construct some exactly solvable models. However, we should mention that the specification on the basis of physical considerations of the behavior of the normalizations in accordance with a definite law makes it possible to select uniquely the potential from the continuous family of equivalent potentials and to obtain in analytic form the corresponding solutions, as has been demonstrated for the example of symmetric potentials transparent with respect to the fast variable.

4.3. Two-dimensional exactly solvable models obtained in the mutually consistent formulation

Since the task of this paper is to investigate exactly solvable models, we represent the algebraic scheme of solution of the many-channel inverse problem in the adiabatic representation, doing this on the basis of the results of Refs. 5 and 22. First, using the technique of degenerate kernels, 1) we determine in explicit analytic form the Bargmann potential matrix $V'(x)$ and its corresponding matrix of solutions of the system of equations (31) with respect to the scattering data $\{S'(p),\{M'_\lambda\},\{E_\lambda\}\}$. 2) We go over from the fixed-basis representation to the representation of the basis (27), which changes from layer to layer, using an ordinary unitary transformation. This makes it possible to find the terms and the normalization functions corresponding to them. 3) Finally, using the algebraic procedure for solving the parametric inverse problem, we determine the two-dimensional potential and the two-dimensional wave functions of the terms. This is a closed procedure of completely self-consistent determination in analytic form of two-dimensional solutions and potentials.

Using the technique of degenerate kernels given above, we give an example of analytic modeling of a two-dimensional problem obtained by analytic solution of the system of slow equations and of the parametric problem. To achieve the greatest simplification, we choose as reference potential $V'(x)$ for recovery of the matrix of the potential interaction the potential $\mathring{V}'(x)=0$. For potentials that are transparent with respect to the slow variable, there remains in $Q^M(x,x')$ only a sum over the bound states:

$$Q_{ij}(x,x')=\sum_\lambda^N \exp(-\varkappa_i^\lambda x)\gamma_i^\lambda\gamma_j^\lambda \exp(-\varkappa_j^\lambda x). \tag{65}$$

From the matrix analog of the basic relations of the inverse problem,

$$K(x,x')+Q(x;x')+\int_{x(0)}^{\infty(x)} K(x,x'')Q(x'',x')dx''=0, \tag{66}$$

$$V(x)=\mathring{V}(x)\mp 2\frac{d}{dy}K(x,x), \tag{67}$$

$$\phi(k,x)=\mathring{\phi}(k,x)+\int_{x(0)}^{\infty(y)} K(x,x')\mathring{\phi}(k,x')dx', \tag{68}$$

we can readily obtain the explicit form for the matrix elements of the potential and solutions:[5,34]

$$V'_{ij}(x)=2\frac{d}{dx}\sum_{\nu\lambda} \exp(-\varkappa_i^\nu x)\gamma_i^\nu P_{\nu\lambda}^{-1}(x)\gamma_j^\lambda \exp(-\varkappa_j^\lambda x), \tag{69}$$

$$F'^{\pm}_{jj'}(k,X)=\exp(\pm ik_jX)\delta_{jj'}$$
$$-\frac{\gamma_j\gamma_{j'}\exp(-\varkappa_jX)\int_X^\infty \exp(-(\varkappa_{j'}\pm ik_{j'})X')dX'}{1+\sum_i^m(\gamma_i^2/2\varkappa_i)\exp(-2\varkappa_iX)}, \tag{70}$$

where

$$P_{\nu\lambda}=\delta_{\nu\lambda}+\sum_{j'}^m \frac{\gamma_{j'}^\nu\gamma_{j'}^\lambda}{\varkappa_{j'}^\nu+\varkappa_{j'}^\lambda}\exp(-(\varkappa_{j'}^\nu+\varkappa_{j'}^\lambda)x).$$

We shall restrict ourselves to considering the example of "transparent" potential matrices with two channels in Marchenko's approach: $i,j=1,2$. From the procedure for diagonalizing the potential matrix V',

$$\mathscr{U}^{-1}(x)V'(x)\mathscr{U}(x)=\mathscr{E}(x),$$

we obtain $\mathscr{U}(x)$, $\mathscr{E}(x)$, and $\delta(x)$:

$$\mathscr{U}(x)=\begin{pmatrix}\cos\delta(x)/2 & \sin\delta(x)/2\\ -\sin\delta(x)/2 & \cos\delta(x)/2\end{pmatrix}, \tag{71}$$

where

$$\delta(x)/2=\int_x A_{12}(x')dx'.$$

It follows from these relations that

$$\begin{pmatrix}\mathscr{E}_1\cos^2\delta/2+\mathscr{E}_2\sin^2\delta/2 & (\mathscr{E}_1-\mathscr{E}_2)\cos\delta/2\sin\delta/2\\ (\mathscr{E}_1-\mathscr{E}_2)\cos\delta/2\sin\delta/2 & \mathscr{E}_1\sin^2\delta/2+\mathscr{E}_2\cos^2\delta/2\end{pmatrix}$$
$$=\begin{pmatrix}V'_{11} & V'_{12}\\ V'_{21} & V'_{22}\end{pmatrix}.$$

As a result, we have

$$\tan 2\delta(x)=\frac{2V'_{21}(x)}{V'_{11}(x)-V'_{22}(x)} \quad \text{or} \quad \sin\delta(x)=\frac{2V'_{21}(x)}{\mathscr{E}_1(x)-\mathscr{E}_2(x)}, \tag{72}$$

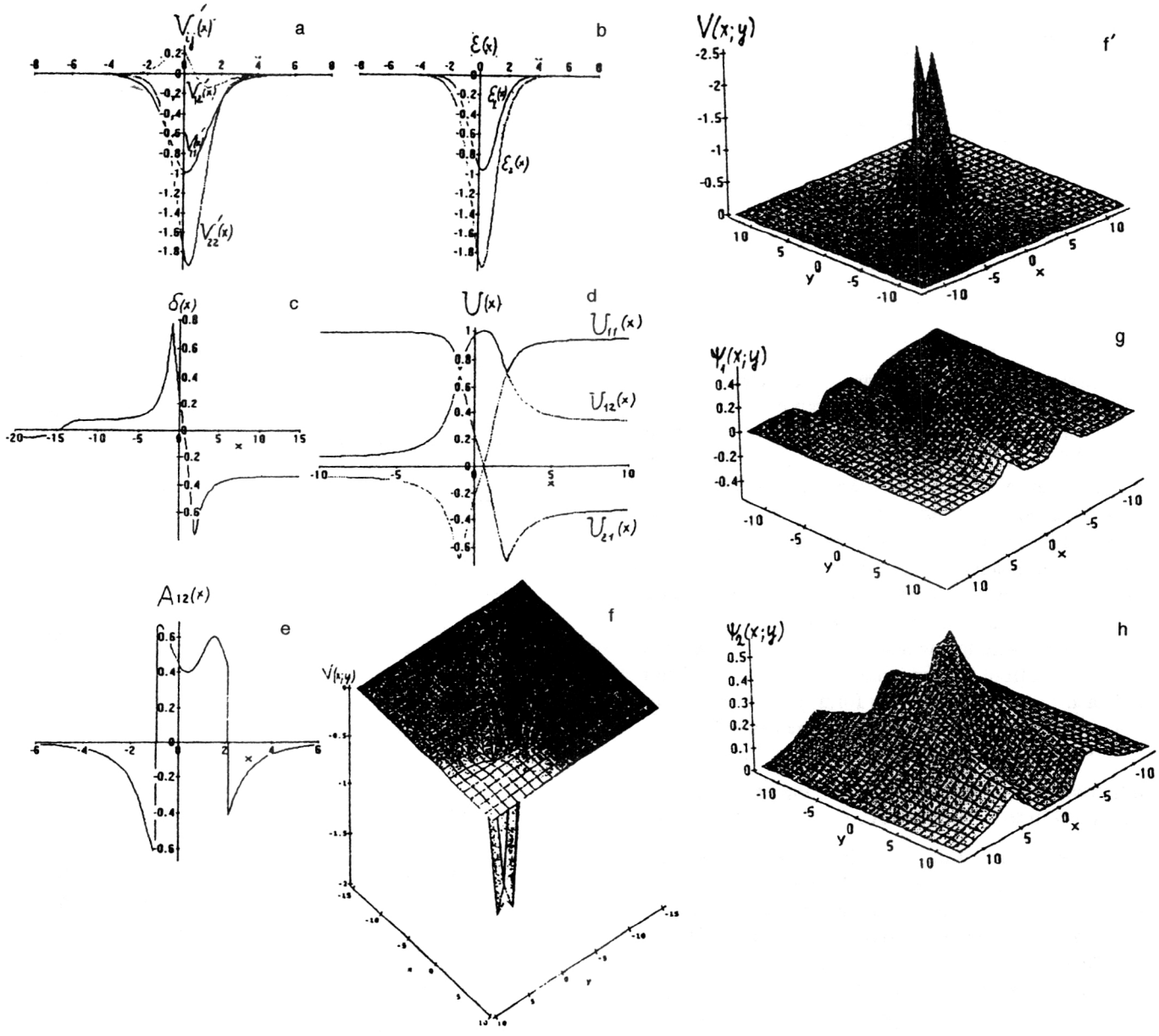

FIG. 12. Mutually consistent recovery of a two-dimensional potential $V(x;y)$ that is transparent with respect to both coordinate variables and symmetric with respect to y by virtue of the choice of the normalizations in the form (43). a) The elements of the transparent potential matrix $V'_{i,j}(x)$ for bound-state energies $E_1=-0.5$, $E_2=-1$ and normalizations $\gamma_1^1=1$, $\gamma_2^2=2$, $\gamma_1^2=0.5$, $\gamma_2^1=0.24$; b) the terms $\mathscr{E}_1(x)$, $\mathscr{E}_2(x)$; c) the phase factor $\delta(x)$; d) the matrix elements of $\mathscr{U}(x)$, $(\mathscr{U}_{11}(x)=\mathscr{U}_{22}(x))$; e) the vector potential $A_{12}(x)$; f) the two-dimensional potential $V(x;y)$; g), h) the corresponding normalized functions $\psi_1(x;y)$ and $\psi_2(x,y)$ of the terms.

$$\mathscr{E}_{1,2}(x)=\frac{1}{2}\left[V'_{11}(x)+V'_{22}(x)\right] \pm \sqrt{(V'_{11}(x)-V'_{22}(x))^2+4V'^2_{12}(x)}]. \qquad (73)$$

We now recover the two-dimensional potential $V(\mathbf{x})$ and determine the two-dimensional wave functions $\psi_i(x;y)$, $i=1,2$, of the parametric problem (14) or (21) that correspond to it, using the obtained spectral data $\{\mathscr{E}_i(x),\gamma_i^2(x)\}$ and, for a fixed value of x, the parametric inverse problem (35), (38), (39). At the same time, for the parametric family of inverse problems the technique of Bargmann potentials makes it possible to construct explicitly the solutions $\psi(x,y)$ and the potential $V(x,y)$ in the framework of the systematic approach using the adiabatic representation. Indeed, the terms are determined uniquely by the relation (73). If we wish to determine potentials that are transparent for all x, we also need to know the normalization, which for potentials transparent and symmetric with respect to y are determined by the values of the energy levels (43). To recover potentials that are transparent but asymmetric with respect to the variable y, it is necessary to use the relations (64) for the normalizations. This will then lead to the recovery of an entire family of transparent potentials—the analog of the phase-equivalent family of potentials in the usual formulation of the inverse problem.

In Figs. 12–15, we give examples of mutually consistent

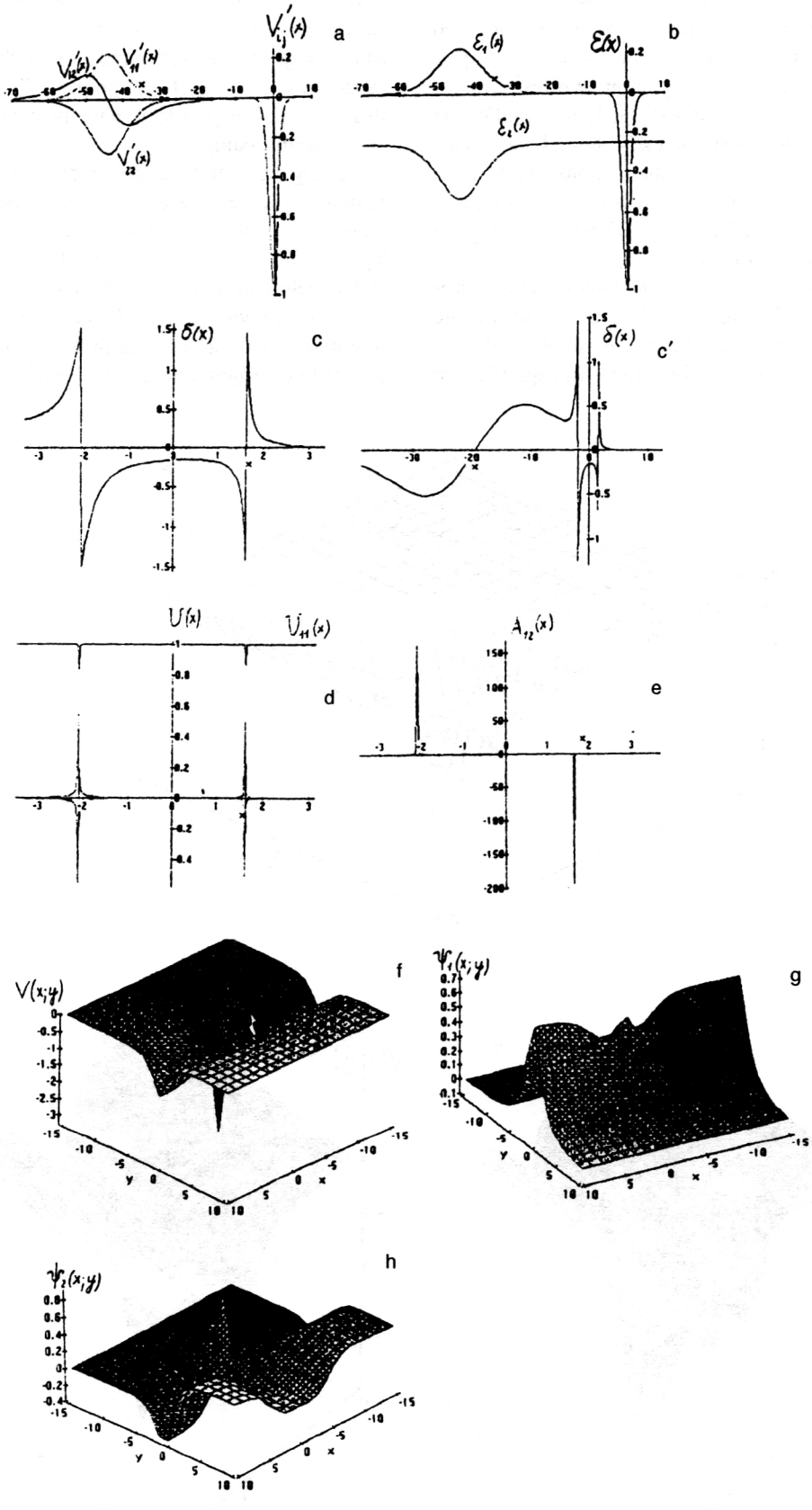

FIG. 13. Mutually consistent recovery of a two-dimensional potential $V(x;y)$ that is symmetric with respect to y, transparent with respect to x and y, and has one bound state $E=-0.5$ and normalizations $\gamma_1^1=1$ and $\gamma_1^2=0.001$. a) The elements $V'_{i,j}(x)$ of the transparent potential matrix; b) the terms $\mathscr{E}_1(x)$, $\mathscr{E}_2(x)$; c) the phase factor $\delta(x)$; d) the matrix elements of $\mathscr{U}(x)$; e) the vector potential $A_{12}(x)$; f) the two-dimensional potential $V(x;y)$; g), h) the corresponding normalized functions $\psi_1(x;y)$ and $\psi_2(x,y)$ of the terms.

analytic solution of the complete problem for a potential that is transparent with respect to both the slow variable x and the fast variable y. Both problems are solved in the modified Marchenko approaches for a two-channel system of equations and parametric Schrödinger equation. We first investigate a system with two bound states (Figs. 12a–12e). For the chosen values of the parameters, namely, bound-state energies $E_1=-0.5$ and $E_2=-1$ and normalizations $\{\gamma_j^\lambda\}$: $\gamma_1^1=1$, $\gamma_2^2=2$, $\gamma_1^2=0.5$, $\gamma_2^1=0.24$, we find a transparent potential matrix, the elements $V'_{11}(x)$, $V'_{12}(x)$, $V'_{21}(x)=V'_{12}(x)$, $V'_{22}(x)$ of which are shown graphically in Fig. 12a. The terms $\mathscr{E}_1(x)$, $\mathscr{E}_2(x)$ calculated by means of the diagonalization procedure (73) are shown in Fig. 12b. Figure 12c shows the phase factor $\delta(x)$, which occurs in the definition of the diagonalization matrix $\mathscr{U}(x)$ (71) (Fig. 12d). In Fig. 12e, we show the vector potential $A(x)$, and, finally, in Figs. 12f, 12g, and 12h we show a two-dimensional potential and the term wave functions corresponding to it. The potential, which is symmetric and transparent with respect to y, has, as we could already see from the preceding examples, a certain structure with respect to y, namely, a main layer and several weak rapidly damped layers.

Using the analytic expressions (69)–(73), we consider an example of a transparent potential matrix with one bound state $\nu=1$ and one threshold: $E_1=-\varkappa_j^2+\varepsilon_j$, $j=1,2$, $E_1=-0.5$, $\varepsilon_1=0$, $\varepsilon_2=0.25$, $\gamma_1^1=1$, $\gamma_2^1=0.001$. The elements of the potential matrix $V'_{ij}(x)$ and the corresponding terms $\mathscr{E}_j(x)$ are shown in Figs. 13a and 13b. The two-dimensional potential $V(x,y)$ and the term wave functions $\psi_1(x;y)$ and $\psi_2(x;y)$ corresponding to it are shown in Figs. 13f, 13g, and

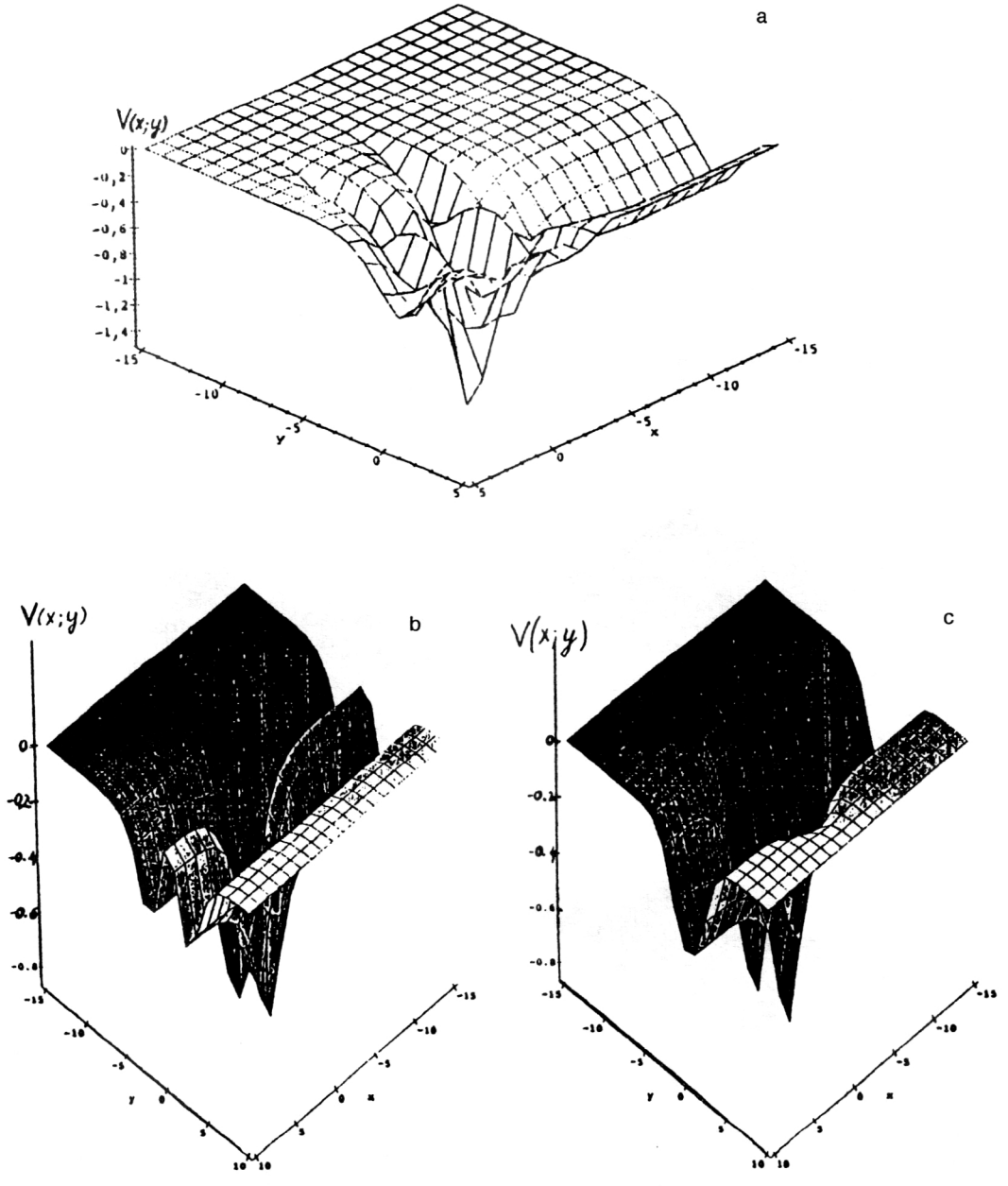

FIG. 14. The family of potentials corresponding to different choices of the normalization functions; a) $V(x,y)$ corresponds to the choice of the normalization in accordance with (65); b)–g) $V(x,y)$ corresponds to a different choice of the fixed basis.

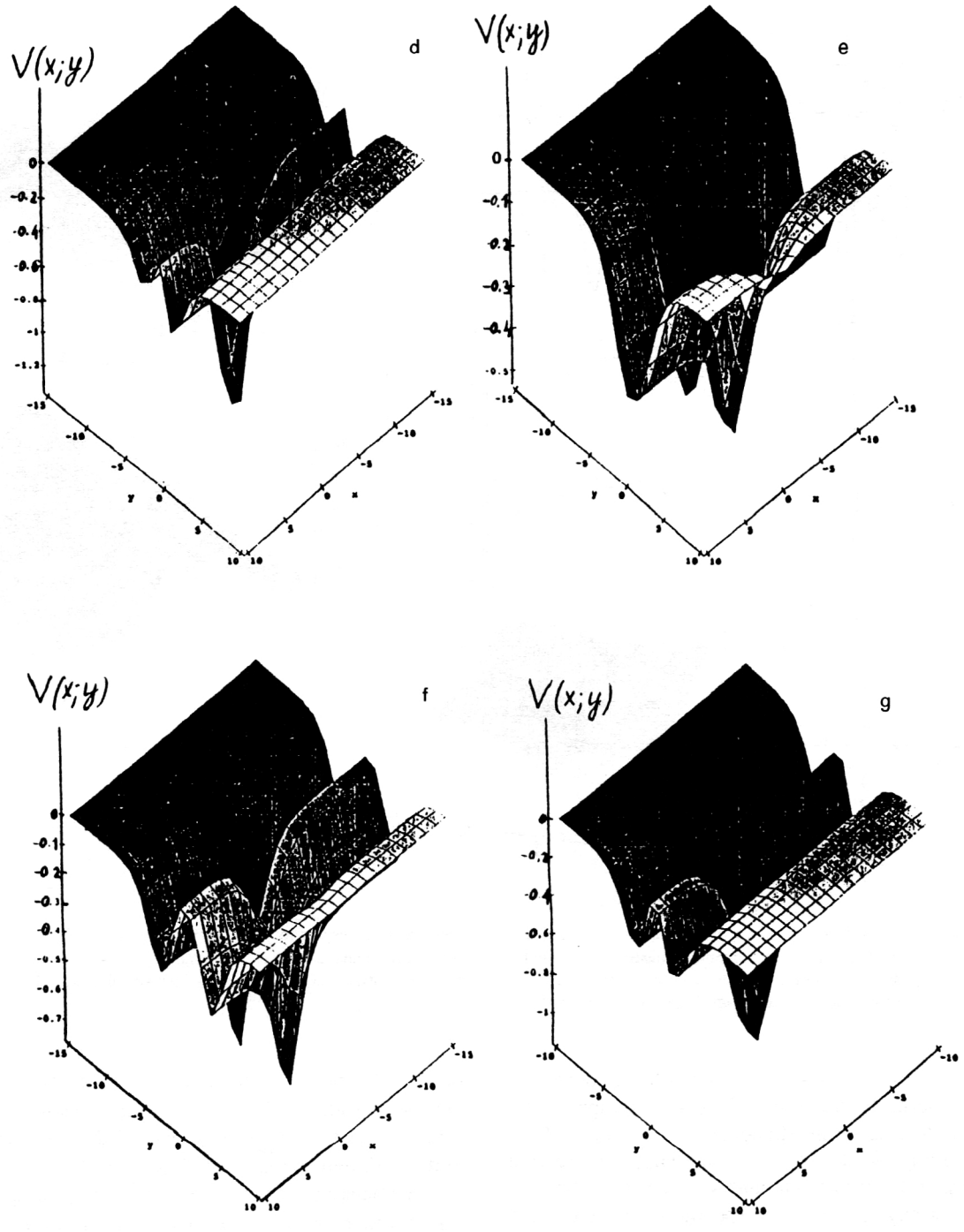

FIG. 14 (*Continued.*)

13h. For comparison, in Figs. 14a–14g, we show a family of two-dimensional potentials for different choices of the fixed basis. The behavior of the matrix elements $A_{12}(x) = -A_{21}(x)$ of the induced vector potential and of the matrix elements $\mathscr{V}_{ij}(x)$ and their argument $\delta(x)$ is demonstrated in Figs. 13c, 13c′, 13d, and 13e. The matrix elements of $A(x)$ are singular at the points of level crossing. At the level crossing, $\delta(x)$ changes by π, corresponding to simultaneous reversal of the sign of the basis functions $\psi(x;y)$ and the functions $F(x)$ of the slow motion. The sign change of the functions is a topological effect that arises in the simple two-channel system. As a result of the first crossing at the point x', we have

$$\begin{pmatrix} \psi_1(x'+\Delta;y) \\ \psi_2(x'+\Delta;y) \end{pmatrix} = -\begin{pmatrix} \psi_1(x'-\Delta;y) \\ \psi_2(x'-\Delta;y) \end{pmatrix};$$

$$\begin{pmatrix} F_1(x'+\Delta) \\ F_2(x'+\Delta) \end{pmatrix} = -\begin{pmatrix} F_1(x'-\Delta) \\ F_2(x'-\Delta) \end{pmatrix}. \tag{74}$$

As a result of the second crossing at the point x'', the sign is restored. Therefore, despite the obvious occurrence of a topological phase as a result of the level crossing, it cannot be detected in our closed physical system with a Bargmann po-

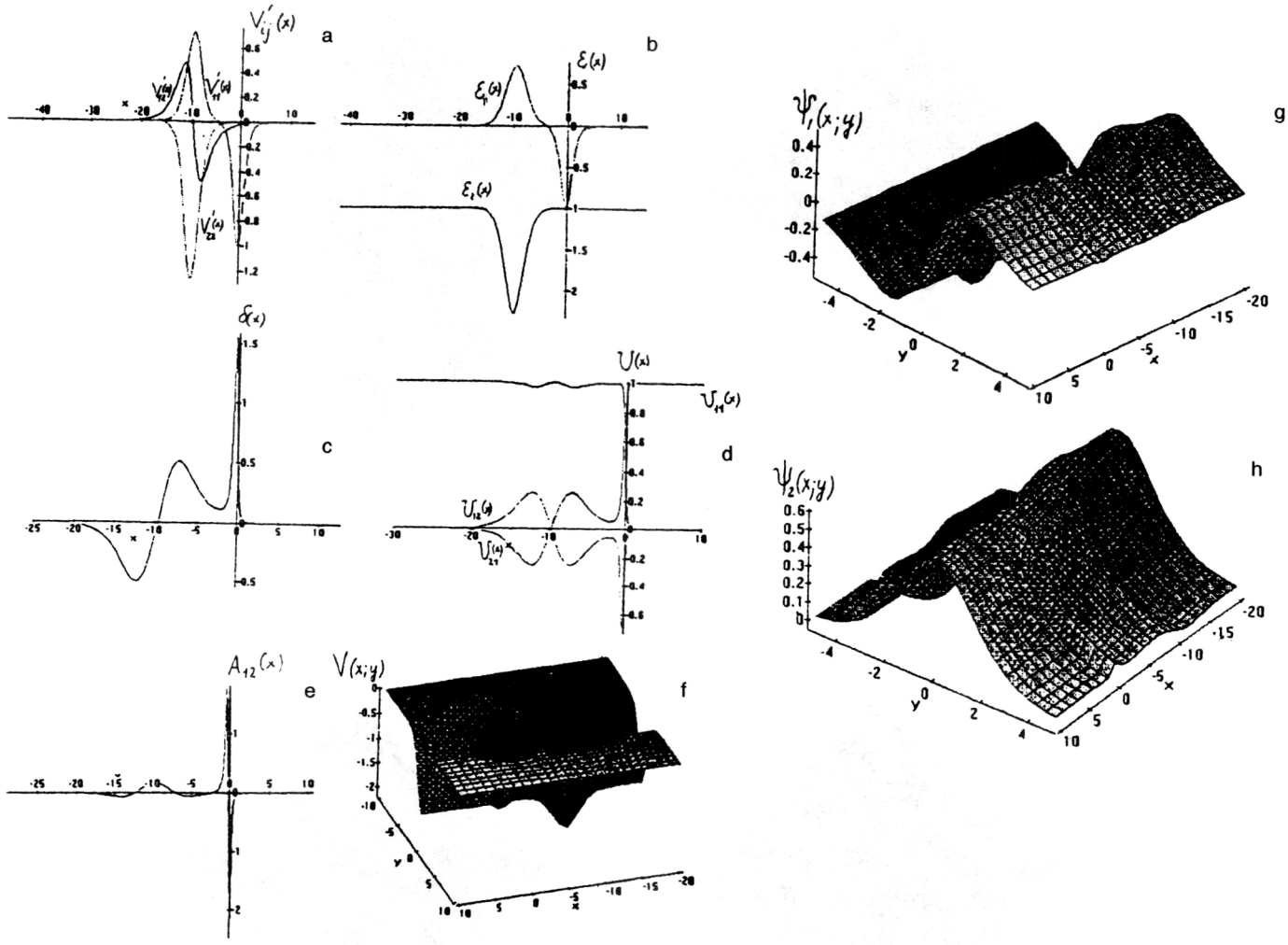

FIG. 15. Mutually consistent recovery of a two-dimensional potential that is symmetric with respect to y and transparent with respect to both coordinate variables and has one bound state $E=-0.5$ and normalizations $\gamma_1^1=1$ and $\gamma_1^2=0.009$. a) The elements $V'_{i,j}(x)$ of the transparent potential matrix; b) the terms $\mathscr{E}_1(x)$, $\mathscr{E}_2(x)$; c) the phase factor $\delta(x)$; d) the matrix elements of $\mathscr{U}(x)$; e) the vector potential $A_{12}(x)$; f) the two-dimensional potential $V(x;y)$; g), h) the corresponding normalized functions $\psi_1(x;y)$ and $\psi_2(x,y)$ of the terms.

tential matrix. However, it may be possible to identify a topological phase in the functions of the slow (respectively, fast) motion in the case of an odd number of crossings. Naturally, one would think of considering the simplest situation with one crossing. However, our analysis has shown that it is not possible, for any choice of the spectral parameters for a two-channel Bargmann potential matrix, to define terms with a single point of degeneracy of the states. A case that demonstrates a situation with two closely spaced but nowhere coincident terms is shown in Fig. 15. It can be seen from Fig. 15c that near the crossing $\delta(x)$ reaches its maximal value but does not pass through zero. It should be noted that term crossing can be achieved only in the presence of thresholds, i.e., intrinsic interaction. From this it may be concluded that topological effects are possible only in sufficiently complicated systems.

The elements of a potential matrix $V'_{ij}(x)$ and the corresponding terms $\mathscr{E}_j(x)$ with one bound state $E_1=-0.5$ and one threshold $\varepsilon=1.0$ and normalizations determined by the values $\gamma_1^1=1$ and $\gamma_2^1=0.009$ are given in Figs. 15a and 15b, respectively. The matrix elements of $\mathscr{U}(x)$, the argument $\delta(x)$, and the vector potential $A(x)$ are shown graphically in Figs. 15c, 15d, and 15e. The two-dimensional potential and the term wave functions $\psi_1(x;y)$ and $\psi_2(x;y)$ corresponding to it are shown in Figs. 15f, 15g, and 15h.

The behavior of the factor $\delta(x)$ is interesting. It turns out that this factor has several maximum values that are not associated with the problem of Landau level crossing.[29] A simple analysis of the expression (72) shows that $\delta(x)$ reaches its maximum value not only when the terms approach each other but also when $2V'_{21}(x)=\mathscr{E}_1(x)-\mathscr{E}_2(x)$, or when $2V'_{21}(x)=V'_{11}(x)-V'_{22}(x)$. Thus, the case with $V'_{12}(x)=0$, $\mathscr{E}_1(x)-\mathscr{E}_2(x)=0$ is special for the situation with $2V'_{12}(x)=V'_{11}(x)-V'_{22}(x)$ and $\delta(x)=(n+1)\pi/2$ for $n=0,2,4,\ldots$. If $\mathscr{E}_1(x)$ is close to $\mathscr{E}_2(x)$ and $V'_{12}(x)$ is equal to zero or to a finite value not equal to $\mathscr{E}_1(x)-\mathscr{E}_2(x)$, then the argument $\delta(x)$ takes the values $\delta(x)=(n+1)\pi/2$ at $n=1,3,5,\ldots$. It is obvious that the inverse problem in the adiabatic approach not only opens up extensive possibilities for constructing exactly solvable models in a space of several dimensions $N\geqslant 2$ but also provides a convenient formalism

for investigating various physical problems, in particular, the problem of level crossing and the geometrical aspects of scattering theory that arise in complicated dynamical systems.

4. CONCLUSIONS

In the framework of the inverse problem in the adiabatic representation, we have constructed some exactly solvable two-dimensional models in the regular and singular cases. We have given some examples of exactly solvable models, using as a basis the developed technique of Bargmann potentials for a parametric family of inverse problems and for systems of equations with a covariant derivative, both in the case of a mutually consistent solution in a systematic approach and on the basis of a solution of the parametric inverse problem alone. Using the formalism of the inverse problem in the Gel'fand—Levitan and Marchenko approaches, we have analyzed various situations possessing both a periodic structure of the terms and nonperiodic analytic behavior of the terms. We have investigated the problem of level crossing on the basis of exactly solvable models of the parametric inverse problem and by means of analysis of two-channel exactly solvable models. We have shown that quantum systems with regular Bargmann potentials acquire a geometric phase in the presence of level crossing even in one-dimensional cases of the slow motion, but such a phase is not manifested in the case of an even number of crossings.

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